MATH 3A WEEK IV
LINEAR TRANSFORMATIONS

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1. Linear Transformations

Definition 1. Let V and W be a vector spaces.

A *linear transformation* from V to W is a function $T : V \to W$ such that

(T1) $T(x + y) = T(x) + T(y)$ for every $x, y \in V$;

(T2) $T(ax) = aT(x)$ for every $a \in \mathbb{R}$ and $x \in V$.

Remark 1. If we say, "let $T : V \to W$ be a linear transformation", but V and W have not yet been specified, it is implicit that V and W are arbitrary vector spaces.

Proposition 1. *Let $T : V \to W$ be a linear transformation. Then $T(0_V) = 0_W$.*

Proof. Since $0 \in \mathbb{R}$, by **(T2)** we have $T(0_V) = T(00_V) = 0T(0_V) = 0_W$. □

Example 1. Let $T : V \to W$ be given by $T(v) = 0_W$ for every $v \in V$. Then T is a linear transformation, called the *zero* transformation.

Example 2. Let $T : V \to V$ be given by $T(v) = av$, where $a \in \mathbb{R}$ is a fixed real number. Then T is a linear transformation, called *dilation by a*.

Example 3. Let V be a vector space and let $X = \{x_1, \dots, x_n\}$ be a basis for V. Then every point $v \in V$ can be written in a unique way as a linear combination from X. Select a subset $Y = \{x_1, \dots, x_k\} \subset X$ and set $W = \operatorname{span}(Y)$; note that W is a vector space, and that Y is a basis for W.

Define a function $T : V \to W$ by $T(v) = \sum_{i=1}^{k} a_i x_i$, where $v = \sum_{i=1}^{n} a_i x_i$. Then T is a linear transformation, called *projection onto W*.

Remark 2. Linear transformations are so named because they take lines to lines (or to a point), planes to planes (or to lines or to a point), and so forth. We now show this.

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2. Transformations and Bases

Proposition 2. *Let $T : V \to W$ be a linear transformation and let $X \subset V$. Then $T(\operatorname{span}(X)) = \operatorname{span}(T(X))$.*

Proof. To show that two sets are equal, we show that each is contained in the other.

Let $w \in T(\operatorname{span}(X))$. Then $w = T(v)$ for some $v \in \operatorname{span}(X)$. Since $v \in \operatorname{span}(X)$, there exist vectors $x_1, \ldots, x_n \in X$ and real numbers $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$v = \sum_{i=1}^{n} a_i x_i.$$

Since T is a linear transformation, it passes through summations and scalar multiplications, so

$$w = T(v) = T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} T(a_i x_i) = \sum_{i=1}^{n} a_i T(x_i).$$

This latter expression is in the span of X, so $w \in \operatorname{span}(X)$.

Let $w \in \operatorname{span}(T(X))$. Then there exist vectors $w_1, \ldots, w_m \in T(X)$ and real numbers $b_1, \ldots, b_m$ such that

$$w = \sum_{i=1}^{m} b_i w_i.$$

For each i, since $w_i \in \operatorname{span}(X)$, the exists $x_i \in X$ such that $w_i = T(x_i)$. This gives

$$w = \sum_{i=1}^{m} b_i w_i = \sum_{i=1}^{m} b_i T(x_i) = \sum_{i=1}^{m} T(b_i x_i) = T(\sum_{i=1}^{m} b_i x_i).$$

Since $\sum_{i=1}^{m} b_i x_i \in \operatorname{span}(X)$, we see that $w \in T(\operatorname{span}(X))$. □

Proposition 3. *Let V and W be vector spaces. Let $X = \{v_1, \ldots, v_n\} \subset V$ be a basis for V. Let $Y = \{w_1, \ldots, w_n\} \subset W$. Then there exists a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$.*

Proof. For each $v \in V$, there exist unique real numbers $a_1, \ldots, a_n$ such that $v = \sum_{i=1}^{n} a_i v_i$. Define $T(v) = \sum_{i=1}^{n} a_i w_i$. It is clear that $T(v_i) = w_i$, and it is easy to verify that T is linear. Uniqueness comes from the necessity of this definition, given that we require T to be linear. □

Corollary 1. *Let $T : V \to W$ be a linear transformation. Then T is completely determined by its effect on any basis for V.*

Remark 3. The above idea is a double edged sword. We completely know a transformation $T : V \to W$ if we know its effect on any basis for V. On the other hand, if we wish to construct a linear transformation, we only need to specify its effect on some basis.

3. Transformations and Subspaces

Proposition 4. *Let $T : V \to W$ be a linear transformation and let $U \leq V$. Then $T(U) \leq W$.*

Proof. We have $T(U) = T(\mathrm{span}(U)) = \mathrm{span}(T(U))$. Thus $T(U)$ is a subspace, since it equals its own span. $\square$

Proposition 5. *Let $T : V \to W$ be a linear transformation and let $U \leq V$. If U is finite dimensional, then $T(U)$ is finite dimensional, and*

$$\dim(T(U)) \leq \dim(U).$$

Proof. Suppose that U is finite dimensional, and let $X \subset U$ be a basis for U. Then $T(U) = T(\mathrm{span}(X)) = \mathrm{span}(T(X))$, so $T(U)$ is spanned by the finite set $T(X)$. If Y is a basis for $T(U)$, then $|Y| \leq |T(X)| \leq |X|$, that is, $\dim(T(U)) \leq \dim(U)$. $\square$

Example 4. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x, y, z) = (x, y)$.
Let U be yz-plane; then $T(U)$ is the y-axis.

Proposition 6. *Let $T : V \to W$ be a linear transformation and let $U_1, U_2 \leq V$. Then $T(U_1 + U_2) = T(U_1) + T(U_2)$.*

Proof. We write this proof as a chain of logical equivalences.

$$\begin{aligned}
w \in T(U_1 + U_2) &\Leftrightarrow w = T(u_1 + u_2) && \text{for some } u_1 \in U_1, u_2 \in U_2 \\
&\Leftrightarrow w = T(u_1) + T(u_2) && \text{because } T \text{ is linear} \\
&\Leftrightarrow w \in T(U_1) + T(U_2) && \text{by definition of image.}
\end{aligned}$$

$\square$

Proposition 7. *Let $T : V \to W$ be a linear transformation and let $U \leq W$. Then $T^{-1}(U) \leq V$.*

Proof. We verify the three properties of a subspace.

(S0) Since $T(0_V) = 0_W \in U$, we see that $0_V \in T^{-1}(U)$.

(S1) Let $v_1, v_2 \in T^{-1}(U)$. Then $T(v_1), T(v_2) \in U$. Thus $T(v_1) + T(v_2) \in U$ because U is a subspace. But $T(v_1) + T(v_2) = T(v_1 + v_2)$ because T is a linear transformation, which shows that $v_1 + v_2 \in T^{-1}(U)$.

(S2) Let $v \in T^{-1}(U)$ and $a \in \mathbb{R}$. Then $T(v) \in U$, so $aT(v) \in U$, whence $T(av) \in U$. Thus $av \in T^{-1}(U)$. $\square$

Example 5. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x, y, z) = (x, y)$. Let $U = \{0\}$. Then $T^{-1}(U)$ is the z-axis.

4. Kernels and Injectivity

Definition 2. Let $T : V \to W$ be a linear transformation.

The *kernel* of T is the subset of V denoted by $\ker(T)$ and defined as

$$\ker(T) = \{v \in V \mid T(v) = 0\}.$$

Remark 4. Note that an alternate way of writing this is $\ker(T) = T^{-1}(0)$.

Proposition 8. *Let $T : V \to W$ be a linear transformation. Then $\ker(T) \leq V$.*

Proof. Since $\{0\} \leq W$, this follows from Proposition 7. □

Proposition 9. *Let $T : V \to W$ be a linear transformation. Then T is injective if and only if $\ker(T) = \{0\}$.*

Proof.

($\Rightarrow$) Suppose that T is injective. Let $v \in \ker(T)$. Then $T(v) = 0_W$; but $T(0_V) = 0_W$, so since T is injective, $v = 0_V$. Thus $\ker(V) = \{0_V\}$.

($\Leftarrow$) Suppose that $\ker(T) = \{0_W\}$. Let $v_1, v_2 \in V$ such that $T(v_1) = T(v_2)$. Then $T(v_1) - T(v_2) = 0_W$, and since T is linear, $T(v_1 - v_2) = 0_W$. Since $\ker(T) = \{0_V\}$, we have $v_1 - v_2 = 0_V$. Thus $v_1 = v_2$ Therefore T is injective. □

Proposition 10. *Let $T : V \to W$ be a linear transformation. Then T is injective if and only if for every independent subset $X \subset V$, $T(X)$ is independent.*

Proof. We prove the contrapositive in both directions.

($\Rightarrow$) Suppose that $X \subset V$ is independent but that $T(X)$ is dependent. Then there exists a nontrivial dependence relation

$$a_1 T(x_1) + \cdots + a_n T(x_n) = 0,$$

where $x_i \in X$ and $a_i \in \mathbb{R}$, not all zero. Then $T(\sum_{i=1}^n a_i x_i) = 0$, so $\sum_{i=1}^n a_i x_i$ is a nontrivial member of $\ker(T)$. Thus T is not injective.

($\Leftarrow$) Suppose that T is not injective. Then its kernel is nontrivial, so there exists an nonzero vector $v \in V$ such that $T(v) = 0$. Since $v \neq 0$, the set $\{v\}$ is independent. But its image $T(v)$ is dependent. □

Proposition 11. *Let $T : V \to W$ be a linear transformation. Let X be a basis for V. Then T is injective if and only if $T(X)$ is a basis for $T(V)$.*

Proof. Suppose X spans V. Then

$$T(V) = T(\mathrm{span}(X)) = \mathrm{span}(T(X)).$$

Now the result follows immediately from the preceding proposition. □

Corollary 2. *Let $T : V \to W$ be an injective linear transformation. Let X be a basis for V. Then*

- **(a)** *$T(X)$ is a basis for $T(V)$;*
- **(b)** $\dim(V) = \dim(T(V))$.

5. Kernels and Cosets

Definition 3. Let V be a vector space and let $W \leq V$.

A *coset* (or "translation") of W is a subset of V of the form

$$x + W = \{x + w \mid w \in W\},$$

where $x \in V$.

Example 6. Let $Z = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0\}$. Then Z is commonly known as the "z-axis". A coset of Z is a set of the form $v + Z$, where $v \in \mathbb{R}^3$. In fact, we can always select v to lie in the xy-plane; we see that $v + Z$ is a vertical line in $\mathbb{R}^3$, parallel to the z-axis, translated away by the vector v.

Proposition 12. *Let V be a vector space and let $W \leq V$. Let $v_1, v_2 \in V$. Then*

- **(a)** $V = \cup_{v \in V}(v + W)$;
- **(b)** $(v_1 + W) \cap (v_2 + W) \neq \varnothing \Rightarrow (v_1 + W) = (v_2 + W)$.

Proof. Exercise. □

Proposition 13. *Let V be a vector space and let $W \leq V$.*
Then $v_1 + W = v_2 + W$ if and only if $v_2 - v_1 \in W$.

Proof. Exercise. □

Proposition 14. *Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation. Let $w \in T(V)$ and let $v \in T^{-1}(w)$. Then*

$$T^{-1}(w) = v + \ker(T);$$

in words, the preimage of w is a coset of the kernel.

Proof. We show that each set is contained in the other.

Let $x \in T^{-1}(w)$. Then $T(x) = w$. Since $v \in T^{-1}(w)$, we have $T(v) = w$. Thus $T(x - v) = T(x) - T(v) = w - w = 0$, so $x - v \in \ker(T)$. Then $x = v + (x - v) \in v + \ker(T)$.

Let $x \in v + \ker(T)$. Then $x = v + k$ where $k \in \ker(T)$, so $T(x) = T(v + k) = T(v) + T(k) = w + 0 = w$, so $x \in T^{-1}(w)$. □

Example 7. A system of m equations in n variables gives a matrix equation

$$Ax = b.$$

The matrix A corresponds to a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$. The solution set of the homogeneous equation $Ax = 0$ is the kernel of T. If v is a particular solution to $Ax = b$, then the solution set is the homogeneous solution offset by the particular solution v.

6. Kernels and Direct Sums

Definition 4. Let V be a vector space and let $U_1, U_2 \leq V$. We say that V is a *direct sum* of U_1 and U_2, and write $V = U_1 \oplus U_2$, if

- **(D1)** $V = U_1 + U_2$;
- **(D2)** $U_1 \cap U_2 = \{0\}$.

Proposition 15. *Let V be a vector space and let X be a basis for V. Let $Y_1 \subset X$ and let $Y_2 = X \smallsetminus Y_1$. Let $U_1 = \operatorname{span}(Y_1)$ and let $U_2 = \operatorname{span}(Y_2)$. Then $V = U_1 \oplus U_2$.*

Proof. We verify the two properties of direct sum.

(D1) We always have $U_1 + U_2 \leq V$; we need to show that $V \subset U_1 + U_2$. If $v \in V$, then V is a linear combination from X because X spans V. Since $X = Y_1 \cup Y_2$, v can be written as a linear combination of some vectors from Y_1 plus a linear combination some vectors from Y_2. Such an element is in $U_1 + U_2$.

(D2) Let $v \in U_1 \cap U_2$. Then v is a linear combination from Y_1 and also v is a linear combination from Y_2. The difference of these is a linear combination from X which equals zero; since X is linearly independent, all of the coefficients must be zero. Thus $v = 0$. □

Proposition 16. *Let V be a vector space. Let $U_1, U_2 \leq V$ such that $V = U_1 \oplus U_2$. Let Y_1 be a basis for U_1 and Y_2 be a basis for U_2. Then $Y_1 \cup Y_2$ is a basis for V.*

Proof. Exercise. □

Corollary 3. *Let V be a finite dimensional vector space and let $U_1, U_2 \leq V$ such that $V = U_1 \oplus U_2$. Then $\dim(V) = \dim(U_1) + \dim(U_2)$.*

Example 8. Let $V = \mathbb{R}^3$. Let U_1 be a plane through the origin in $\mathbb{R}^3$ and let U_2 be a line through the origin in $\mathbb{R}^3$. Then $V = U_1 \oplus U_2$ if and only if the line U_2 does not lie on the plane U_1.

Proposition 17. *Let $T : V \to W$ be a linear transformation. Let $K = \ker(T)$. Then*

- **(a)** *there exists $U \leq V$ such that $V = K \oplus U$;*
- **(b)** *$T \restriction_U : U \to W$ is injective.*

Proof. Let Y_1 be a basis for K and let X be a completion of Y_1 to a basis for X. Let $Y_2 = X \smallsetminus Y_1$. Let $U = \operatorname{span}(Y_2)$. Then by Proposition 15, $V = K \oplus U$. This proves **(a)**.

Recall that $T \restriction_U : U \to W$ is the restriction of T to the set U; that is, we only consider what T does to elements of U. Let $u \in \ker(T \restriction_U)$. Then $T(u) = 0$, so $u \in K$. Thus $u \in K \cap U = \{0\}$, so $u = 0$. Thus the kernel of $T \restriction_U$ is trivial, so $T \restriction_U$ is injective by Proposition 9. □

7. Rank and Nullity

Definition 5. Let V be a finite dimensional vector space and let $T : V \to W$ be a linear transformation. Let $\mathrm{img}(T) = T(V)$ denote the image of T.

The *rank* of T is the dimension of the image of T: $\mathrm{rank} = \dim(\mathrm{img}(T))$.

The *nullity* of T is the dimension of the kernel of T: $\mathrm{nullity} = \dim(\ker(T))$.

Theorem 1. (Rank plus Nullity Theorem)
Let V be a finite dimensional vector space and let $T : V \to W$ be a linear transformation. Then $\dim(V) = \dim(\ker(T)) + \dim(\mathrm{img}(T))$.

Proof. Let $K = \ker(T)$. By Proposition 17 **(a)**, there exists a subspace $U \leq V$ such that $V = K \oplus U$. Thus $\dim(V) = \dim(K) + \dim(U)$. By Proposition 17 **(b)**, the linear transformation $T \restriction_U : U \to W$ is injective, so $\dim(T(U)) = \dim(U)$. Thus

$$\dim(V) = \dim(K) + \dim(U) = \dim(\ker(T)) + \dim(\mathrm{img}(T)).$$

$\square$

Corollary 4. *Let V and W be a finite dimensional vector spaces of the same dimension. Let $T : V \to V$ be a linear transformation. Then T is injective if and only if T is surjective.*

Proof. Exercise. $\square$

8. Composition of Linear Transformations

Proposition 18. *Let $S : U \to V$ and $T : V \to W$ be linear transformations. Then $T \circ S : U \to W$ is a linear transformation.*

Proof. We verify the two properties of a linear transformation.

(T1) Let $u_1, u_2 \in U$. Then

$$T(S(u_1 + u_1)) = T(S(u_1) + S(u_2)) = T(S(u_1)) + T(S(u_2)).$$

(T2) Let $u \in U$ and $a \in \mathbb{R}$. Then

$$T(S(au)) = T(aS(u)) = aT(S(u)).$$

□

Example 9. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be dilation by a and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be dilation by b. Then $T \circ S : \mathbb{R}^2 \to \mathbb{R}^3$ is dilation by ab.

Example 10. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by α degrees and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by β degrees. Then $T \circ S : \mathbb{R}^2 \to \mathbb{R}^3$ is rotation by $\alpha + \beta$ degrees.

Definition 6. Let $T : V \to W$ be a linear transformation.

We say that T is *invertible* if there exists a linear transformation $S : W \to V$ such that $S \circ T = \mathrm{id}_V$ and $T \circ S = \mathrm{id}_W$. Such an S is called the *inverse* of T; it is necessarily unique, and is denoted by T^{-1}.

Proposition 19. *Let $T : V \to W$ be a linear transformation.*
Then T is invertible if and only if T is bijective.

Proof. Exercise. □

Proposition 20. *Let $T : U \to V$ be a linear transformation.*
Let $S : V \to W$ be an injective linear transformation.
Then $\ker(S \circ T) = \ker(T)$.

Proof. Exercise. □

Remark 5. Let A be an $m \times n$ matrix and consider the matrix equation $Ax = 0$, where 0 is the zero $n \times 1$ column vector. The solution to this equation is the kernel of the corresponding linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

Let B be A in reduced row echelon form. Row reduction of A corresponds to warping m-space by invertible transformations. Then $\ker(T_B) = \ker(T_A)$, because $B = UA$, where U is a product of elementary invertible matrices and so it is invertible; then T_U is injective. Therefore $\ker(T_B) = \ker(T_U \circ T_A) = \ker(T_A)$.

Moreover, the basic columns of B are clearly linearly independent. Then the pullback of these basic columns via U^{-1} gives linearly independent vectors in $\mathrm{img}(T) = T_A(\mathbb{R}^n)$, the image of T_A.

A basis for the kernel of T_A is given by modifying the free columns of B in the manner prescribed in solving $Ax = 0$.

A basis for the image of T_A is given by the columns of A corresponding to the basic columns of B.

Example 11. Let $e_1, \ldots, e_4$ be the standard basis vectors for $\mathbb{R}^4$. Let

$$v_1 = (2, -4, 4), v_2 = (1, -1, 3), v_3 = (3, -7, 5), v_4 = (0, 2, 5) \in \mathbb{R}^3.$$

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the unique linear transformation given by $T(e_i) = v_i$. Find a basis for the image and the kernel of T.

Solution. Set

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ -4 & -1 & -7 & 2 \\ 4 & 3 & 5 & 5 \end{bmatrix}.$$

Row reduce A; the corresponding reduced row echelon matrix is

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The basic variables are x_1, x_2, and x_4. The free variable is x_3. So the solution to $Ax = 0$ is

$$x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix};$$

thus $\{(-2, 1, 1, 0)\}$ is a basis for $\ker(T)$, and $\{(2, -4, 4), (1, -1, 3), (0, 2, 5)\}$ is a basis for $\mathrm{img}(T)$, the image of T. □

Remark 6. Let $Y = \{v_1, \ldots, v_n\} \in \mathbb{R}^m$. We wish to determine whether or not the set Y is independent. If $n > m$, we know they cannot be independent, so assume that $n \leq m$.

Form the matrix $A = [v_1 \mid \cdots \mid v_n]$. Corresponding to A is a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$. We know that $n = \dim(\mathbb{R}^n) = \dim(\ker(T_A)) + \dim(\mathrm{img}(T_A))$. Now X is independent if and only if there span in $\mathbb{R}^m$ is a vector space of dimension n. This span is exactly $\mathrm{img}(T_A)$. Thus X is independent if and only if $\dim(\mathrm{img}(T_A)) = n$. This is the case if and only if $\dim(\ker(T_A)) = 0$.

Row reduce A to obtain a matrix B; only forward elimination is necessary. Now X is dependent if and only if B has a free column, which is the case if and only if B has a zero row (since $n \leq m$).

9. Isomorphisms

Definition 7. Let V and W be vector spaces.

An *isomorphism* from V to W is a bijective linear transformation $T : V \to W$. We say that V is *isomorphic* to W, and write $V \cong W$, if there exists an isomorphism $T : V \to W$.

Proposition 21. *Let V be a vector space.*
Then $\mathrm{id}_V : V \to V$ is an isomorphism.

Proof. Clear. □

Proposition 22. *Let $T : V \to W$ be an isomorphism.*
Then $T^{-1} : W \to V$ is an isomorphism.

Proof. Since T is bijective, $T^{-1} : W \to V$ is a function. We verify the properties of a linear transformation.

(T1) Let $w_1, w_2 \in W$. Since T is bijective, there exist unique elements $u_1, u_2 \in U$ such that $T(u_1) = w_1$ and $T(u_2) = w_2$. Now $T(u_2 + u_2) = T(u_1) + T(u_2) = w_1 + w_2$, so $T^{-1}(w_1 + w_2) = u_1 + u_2 = T^{-1}(w_1) + T^{-1}(w_2)$.

(T2) Let $w \in W$ and $a \in \mathbb{R}$. There exists a unique element $u \in U$ such that $T(u) = w$. Then $T(au) = aT(u) = aw$, so $T^{-1}(aw) = au = aT^{-1}(w)$. □

Proposition 23. *Let $S : U \to V$ and $T : V \to W$ be isomorphisms.*
Then $T \circ S : U \to W$ is an isomorphism.

Proof. We have seen that the composition of linear transformations is linear, and we always have that the composition of bijective functions is bijective. □

Remark 7. Let U, V, and W be vector spaces. Then

- **(a)** $V \cong V$;
- **(b)** $V \cong W \Leftrightarrow W \cong V$;
- **(c)** $U \cong V$ and $V \cong W \Rightarrow U \cong W$.

This says that isomorphism is an *equivalence relation*.

Proposition 24. *Let $T : V \to W$ be a linear transformation. Let X be a basis for V. Then T is an isomorphism if and only if $T(X)$ is a basis for W.*

Proof.

($\Rightarrow$) Suppose that T is an isomorphism. Then T is injective, so by Proposition 11, $T(X)$ is a basis for $T(V)$. But T is also surjective, so $T(V) = W$, and the result follows.

($\Leftarrow$) Suppose that $T(X)$ is a basis for W. Then T is clearly surjective, and by Proposition 11, T is also injective. Thus T is an isomorphism. □

Remark 8. In light of Proposition 3, we may construct an isomorphism between spaces by sending a basis to a basis.

Definition 8. Let V be a finite dimensional vector space of dimension n.

An *ordered basis* for V is an ordered n-tuple $(x_1, \dots, x_n) \in V^n$ of linearly independent vectors from V.

Remark 9. Note that if $(x_1, \dots, x_n)$ is an ordered basis, then $X = \{x_1, \dots, x_n\}$ is a basis. With this understanding, we may say: "let X be an ordered basis", by which we mean that X is the basis which corresponds to an ordered basis.

Theorem 2. *Let V be a finite dimensional vector space of dimension n. Let $X = \{x_1, \dots, x_n\}$ be an ordered basis for V. Define a linear transformation*

$$\Gamma_X : V \to \mathbb{R}^n \quad \text{by} \quad \Gamma_X(x_i) = e_i.$$

Then Γ_X is an isomorphism.

Description. We have already essentially proven this, so let us describe it in more detail.

Every element of V may be written *in a unique way* as a linear combination of elements from X: if $v \in V$, then $v = \sum_{i=1} a_i x_i$ for some real number $a_1, \dots, a_n$. Then

$$\Gamma_X(v) = \sum_{i=1}^{n} a_i \Gamma_X(x_i) = \sum_{i=1}^{n} \sum_{i=1}^{n} a_i e_i = (a_1, \dots, a_n);$$

this is the linear transformation that sends the basis X of V to the standard basis for $\mathbb{R}^n$, whose existence, uniqueness, and linearity is guaranteed by Proposition 3. It is an isomorphism by Proposition 24. □

Corollary 5. *Let V and W be vector spaces of dimension n. Then $V \cong W$.*

Proof. Every finite dimensional vector space has a basis. Let X be an ordered basis for V and let Y be an ordered basis for W. Since $\Gamma_Y : W \to \mathbb{R}^n$ is an isomorphism, it is invertible, and its inverse is also an isomorphism. Since the composition of isomorphisms is an isomorphism, we see that

$$\Gamma_Y^{-1} \circ \Gamma_X : V \to W$$

is an isomorphism, so $V \cong W$. □

Remark 10. Even though two vector spaces of the same dimension are isomorphic, there are many *ways* in which they are isomorphic. Indeed, each basis X for V gives a *different* isomorphism $\Gamma_X : V \to \mathbb{R}^n$. Controlling this is one of the challenges of linear algebra.

10. Computing Linear Transformations via Matrices

Remark 11. Let V be a vector space of dimension n and let W be a vector space of dimension m. Let $T : V \to W$ be a linear transformation. If we know a basis for V and for W, we can use matrices to compute information about T.

Let X be a basis for V and let Y be a basis for W. Then $\Gamma_X : V \to \mathbb{R}^n$ is an isomorphism and $\Gamma_Y : W \to \mathbb{R}^m$ is an isomorphism. These isomorphisms pick off the coefficients of any vector in V and W and allow us to think of them as vector in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Actually, what we are doing is defining a transformation $S : \mathbb{R}^n \to \mathbb{R}^m$ given by $S = \Gamma_Y \circ T \circ \Gamma_X$. In this case,

$$T = \Gamma_Y^{-1} \circ S \circ \Gamma_X.$$

This can be written in diagram form:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Gamma_X"'] & W \arrow[d, "\Gamma_Y"] \\ \mathbb{R}^n \arrow[r, "S"'] & \mathbb{R}^m \end{tikzcd}$$

This says that to compute $T(v)$, it suffices to push v into $\mathbb{R}^n$ via $u = \Gamma_X(v)$, compute $S(u)$, then pull this result back to W via Γ_Y.

But $S : \mathbb{R}^n \to \mathbb{R}^m$ corresponds to a matrix A, and we can compute Au by matrix multiplication. This also allows us to compute kernels, images, and so forth via matrices.

Example 12. Let $v_1 = (1, 0, 0, 0), v_2 = (1, 0, 1, 0), v_3 = (1, 0, 0, 1) \in \mathbb{R}^4$. Let V be the subspace of $\mathbb{R}^4$ spanned by $\{v_1, v_2, v_3\}$; these form a basis for V. Let $W = \mathbb{R}^2$ Let $w_1 = (1, 2), w_2 = (-1, 0), w_3 = (3, 2) \in W$. Let $T : V \to W$ be the unique linear transformation given $T(v_i) = w_i$. Find a basis for the kernel of T.

Solution. Let e_1, e_2, e_3 be the standard basis vectors for $\mathbb{R}^3$. Let $S : V \to \mathbb{R}^3$ be given by $T(v_i) = e_i$. Then S is an isomorphism. Let $R : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(e_i) = w_i$. The matrix for R is

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 2 \end{bmatrix}.$$

Row reduce A to get

$$UA = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}.$$

The kernel of R is spanned by the vector $(-7, -4, 1)$.

Now $T = S^{-1}RS$. Thus $ST = RS$. Then

$$\ker(T) = \ker(ST) = \ker(RS) = S^{-1}(\ker(R)).$$

Thus to find $\ker(T)$, pull the vector $(-7, -4, 1)$ back through S (find its preimage). This is $-7(1, 1, 0, 0) - 4(1, 0, 1, 0) + (1, 0, 0, 1) = (-10, -7, -4, 1)$. The kernel of T is the span of this vector. $\square$

11. Vector Space of Linear Transformations

Proposition 25. *Let V and W be vector spaces and set*

$$\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is linear}\}.$$

Let $S : V \to W$ and $T : V \to W$ be linear transformations. Let $a \in \mathbb{R}$. Define the sum $S+T$ *and the* scalar product aT *by their effect on any vector $v \in V$:*

- $(S+T)(v) = S(v) + T(v)$;
- $(aT)(v) = aT(v)$.

Then

(a) *$S+T : V \to W$ and $aT : V \to W$ are linear transformations;*
(b) *$\mathcal{L}(V,W)$ is a vector space.*

Reason. The verification that $S+T$ and aT are linear transformations is straightforward.

The proof that $\mathcal{L}(V,W)$ is a vector space comes down to the fact that all of the properties **(V1)** through **(V8)** of the vector space W work pointwise on functions into W. $\square$

Remark 12. The vector space $\mathcal{M}_{m\times n}$ of $m \times n$ matrices is isomorphic to $\mathbb{R}^{mn}$, as one expects. But also, we know that matrices correspond to linear transformations of cartesian spaces; we now describe this correspondence in terms of isomorphism.

Proposition 26. *Let $T_{ij} : \mathbb{R}^n \to \mathbb{R}^m$ be given by $T_{ij}(e_j) = e_i$ and $T_{ij}(e_k) = 0$ if $k \neq j$. Then $\{T_{ij} \mid i = 1,\dots,m;\ j = 1,\dots,n\}$ is a basis for $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$.*

Reason. One can show that this set is linearly independent and spans. $\square$

Proposition 27. *Define a function*

$$\Omega_{m\times n} : \mathcal{L}(\mathbb{R}^n,\mathbb{R}^m) \to \mathcal{M}_{m\times n} \quad \text{by} \quad \Omega_{m\times n}(T) = A_T,$$

where $A_T = [T(e_1) \mid \cdots \mid T(e_n)]$ is the matrix corrsponding to a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Then $\Omega_{m\times n}$ is an isomorphism.

Reason. The function $\Omega_{m\times n}$ sends the basis $\{T_{ij}\}$ for $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ to the basis $\{M_{ij}\}$ for $\mathcal{M}_{m\times n}$. $\square$

Proposition 28. *Let V and W be finite dimensional vector spaces. Let $X = \{x_1,\dots,x_n\}$ be an ordered basis for V and $Y = \{y_1,\dots,y_m\}$ be an ordered basis for W. Define a function*

$$\Omega_{Y,X} : \mathcal{L}(V,W) \to \mathcal{M}_{m\times n} \quad \text{by} \quad \Omega_{Y,X}(T) = A_S,$$

where $S = \Gamma_Y \circ T\Gamma_X^{-1}$ and $A_S = [S(e_1) \mid \cdots \mid S(e_n)]$ is the matrix corresponding to S. Then $\Omega_{Y,X}$ is an isomorphism.

Reason. In a manner similar to the case where $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, one can find a basis for $\mathcal{L}(V,W)$ that is sent by $\Omega_{Y,X}$ to the basis $\{M_{ij}\}$ for $\mathcal{M}_{m\times n}$. $\square$

12. Linear Operators

Definition 9. Let V be a vector space.

A *linear operator* on V is a linear transformation $T : V \to V$.

Let $\mathcal{L}(V)$ denote the set of all linear operators on V.

Let V be a vector space and let $S, T : V \to V$ be a linear operators. Then the composition $T \circ S : V \to V$ is a linear operator. Let us drop the $\circ$ from the notation and think of composition of linear operators as multiplication in the set $\mathcal{L}(V)$: thus TS is the transformation $T \circ S$.

This multiplication distributes over addition of operators:

$$T(S + R) = TS + TR; \quad (T + S)R = TR + SR.$$

Thus $\mathcal{L}(V)$ is a set which comes equipped with two operations, addition of transformations and multiplication of transformations. The additive identity of this set is the zero transformation (which we denote by 0), and the multiplicative identity is the identity transformation id_V, which we now denote by 1. Every transformation T has an additive inverse $-T$. A transformation T has a multiplicative inverse T^{-1} if and only if T has a trivial kernel.

Let $a \in \mathbb{R}$. Define $N_a : V \to V$ to be dilation by a: $N_a(v) = av$ for all $v \in V$. Then N_a is a linear operator. Note that N_a commutes with any other operator:

$$N_a T = T N_a.$$

Also note that $N_a T$ is exactly the transformation which we previously described by aT. When N_a occurs on the left, we drop the N from the notation, and simply write aT instead of $N_a T$.

Let $T^2 = TT$, $T^3 = TTT$, and in general, let T^n denote the composition of T with itself n times. This is T applied to the space V over and over. For example, if T is rotation of $\mathbb{R}^2$ by an angle of 45 degrees, then T^4 is rotation by 180 degrees and T^8 is the identity tranformation $I = \mathrm{id}_V$.

Let $T : V \to V$ be a linear operator. We see that any polynomial in T

$$L = T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0$$

is a linear operator. Its effect on $v \in V$ is given by distributing v into the polynomial:

$$L(v) = T^n(v) + a_{n-1}T^{n-1}(v) + \cdots + a_1 T(v) + a_0.$$

13. Linear Algebra and Differential Equations

Consider the differential equation

$$y'' + by' + cy = g(t),$$

where $b, c \in \mathbb{R}$ and $g(t)$ is a smooth function on some open interval $I \subset \mathbb{R}$. To solve this differential equation means to find all smooth functions y such that the function $y'' + by' + cy$ is equal to the function $g(t)$. We use linear algebra to analyse this situation.

Let $I \subset \mathbb{R}$ be an open interval and let $\mathcal{D}(I)$ be the set of smooth real valued functions defined by I; this is a vector space under addition and scalar multiplication of functions. Define $D : \mathcal{D}(I) \to \mathcal{D}(I)$ by $D(f) = f'$, the derivative of f. Then D is a linear transformation. Any polynomial in D is also a linear transformation, called a *differential operator*. Note that the kernel of D is the set of all constant functions on I. This is a one dimensional subspace of $\mathcal{D}(I)$, spanned by the function $f(t) = 1$.

Let $b, c \in \mathbb{R}$ and let $g \in \mathcal{D}(I)$. Define a function

$$L : \mathcal{D}(I) \to \mathcal{D}(I) \quad \text{by} \quad L[y] = y'' + by' + cy.$$

Then L is a differential operator:

$$L = D^2 + bD + c.$$

The general solution to the differential equation

$$y'' + by' + cy = g(t)$$

is of the form $y = y_h + y_p$, where y_h is the general solution to the homogeneous differential equation $L[y] = 0$ and y_p is a particular solution to the differential equation $L[y] = g(t)$. This comes from the fact that the solution to the homogeneous equation is the kernel of L, and the solution to the nonhomogeneous equation is a coset of this kernel.

One may attempt to solve the homogeneous differential equation $L[y] = 0$ by factoring the linear operator L:

$$L = (D - r_1)(D - r_2),$$

where $r_i = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$ are the roots of the polynomial L. Now any solution to $(D - r_2)[y] = 0$ is also a solution to $L[y] = 0$, since $(D - r_1)[0] = 0$. Since $D - r_1$ and $D - r_2$ commute, the same can be said about solutions to $(D - r_1)[y]$. But this differential equation is very easy to solve:

$$(D - r)[y] = 0 \Leftrightarrow y' = ry \Leftrightarrow \log y = r + C \Leftrightarrow y = ke^{rt},$$

where $k = e^C$ is an arbitrary constant of integration. One can show that $\ker(L) = \operatorname{span}\{e^{r_1 t}, e^{r_2 t}\}$.

14. Exercises

Exercise 1. Let V be a vector space.
Let $U_1, U_2 \leq V$ such that $V = U_1 \oplus U_2$. Let Y_1 be a basis for U_1 and Y_2 be a basis for U_2. Show that $Y_1 \cup Y_2$ is a basis for V.

Exercise 2. Let V be a vector space and let $W \leq V$. Let $v_1, v_2 \in V$.

- **(a)** Show that $V = \cup_{v \in V}(v + W)$.
- **(b)** Show that $(v_1 + W) \cap (v_2 + W) \neq \varnothing \Rightarrow (v_1 + W) = (v_2 + W)$.

Exercise 3. Let V be a vector space and let $W \leq V$.
Show that $v_1 + W = v_2 + W$ if and only if $v_2 - v_1 \in W$.

Exercise 4. Let V be a finite dimensional vector space.
Let $U \leq V$ and let $T : V \to V$ be a linear transformation.

- **(a)** Show that $U = V$ if and only if $\dim(U) = \dim(V)$.
- **(b)** Show that T is injective if and only if T is surjective.

Exercise 5. Let $T : V \to W$ be a linear transformation.
Show that T is invertible if and only if T is bijective.

Exercise 6. Let $T : U \to V$ be a linear transformation.
Let $S : V \to W$ be an injective linear transformation.
Show that $\ker(S \circ T) = \ker(T)$.

Exercise 7. Let $T : V \to W$ be a linear transformation and let $U_1, U_2 \leq V$. In each case, prove or give a counterexample.

- **(a)** $T(U_1 \cap U_2) = T(U_1) \cap T(U_2)$;
- **(b)** $V = U_1 \oplus U_2 \Rightarrow T(V) = T(U_1) \oplus T(U_2)$.

Exercise 8. Let $T : V \to W$ be a linear transformation and let $U_1, U_2 \leq W$. In each case, prove or give a counterexample.

- **(a)** $T^{-1}(U_1 \cap U_2) = T^{-1}(U_1) \cap T^{-1}(U_2)$;
- **(b)** $W = U_1 \oplus U_2 \Rightarrow T^{-1}(W) = T^{-1}(U_1) \oplus T^{-1}(U_2)$.

Exercise 9. Let $\mathcal{P}_n$ denote the vector space of polynomial functions of degree less than or equal to n with real coefficients:

$$\mathcal{P}_n = \{f(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}\}.$$

Let $\Gamma : \mathcal{P}_4 \to \mathbb{R}^5$ be given by $\Gamma(x^{i-1}) = e_i$ for $i = 1, \ldots, 5$.
Let $D : \mathcal{P}_4 \to \mathcal{P}_4$ be given by $D(f) = \frac{df}{dx}$.
Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be given by $T = \Gamma \circ D \circ \Gamma^{-1}$.

- **(a)** Describe why Γ is an isomorphism.
- **(b)** Find the matrix corresponding to the linear transformation T.
- **(c)** Find a basis for the image and the kernel of T.
- **(d)** Find a basis for the image and the kernel of D.

Exercise 10. Let $\mathcal{D}(\mathbb{R})$ denote the set of all smooth functions on $\mathbb{R}$.
Let $D : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ be given by $D(f) = \frac{df}{dx}$.
Let $D^n : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ denote D composed with itself n times.
Find $\ker(D^n)$; justify your answer.

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